

ARITHMETIC OF CURVES OVER TWO DIMENSIONAL LOCAL FIELD

BELGACEM DRAOUIL

ABSTRACT. We study the class field theory of curve defined over two dimensional local field. The approach used here is a combination of the work of Kato-Saito, and Yoshida where the base field is one dimensional

1. INTRODUCTION

Let k_1 be a local field with finite residue field and let X be a proper smooth geometrically irreducible curve over k_1 . To study the fundamental group $\pi_1^{ab}(X)$, Saito in [8], introduced the groups $SK_1(X)$ and $V(X)$ and construct the maps $\sigma : SK_1(X) \rightarrow \pi_1^{ab}(X)$ and $\tau : V(X) \rightarrow \pi_1^{ab}(X)^{géo}$ where $\pi_1^{ab}(X)^{géo}$ is defined by the exact sequence

$$0 \rightarrow \pi_1^{ab}(X)^{géo} \rightarrow \pi_1^{ab}(X) \rightarrow Gal(k_1^{ab}/k_1) \rightarrow 0$$

The most important results in this context are:

- 1) The quotient of $\pi_1^{ab}(X)$ by the closure of the image of σ and the cokernel of τ are both isomorphic to $\widehat{\mathbb{Z}}^r$ where r is the rank of the curve.
- 2) For this integer r , there is an exact sequence

$$(1.2) \quad 0 \rightarrow (\mathbb{Q}/\mathbb{Z})^r \rightarrow H^3(K, \mathbb{Q}/\mathbb{Z}(2)) \rightarrow \bigoplus_{v \in P} \mathbb{Q}/\mathbb{Z} \rightarrow \mathbb{Q}/\mathbb{Z} \rightarrow 0$$

where $K = K(X)$ is the function field of X and P designates the set of closed points of X .

These results are obtained by Saito in [8] generalizing the previous work of Bloch where he is reduced to the good reduction case [8, Introduction]. The method of Saito depends on class field theory for two-dimensional local ring having finite residue field. He shows these results for general curve except for the p -primary part in $\text{char } k = p > 0$ case [8, Section II-4]. The remaining p -primary part had been proved by Yoshida in [11].

There is another direction for proving these results pointed out by Douai in [3]. It consists to consider for all l prime to the residual characteristic, the group $\text{Coker } \sigma$ as the dual of the group W_0 of the monodromy weight filtration of $H^1(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$

$$H^1(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) = W_2 \supseteq W_1 \supseteq W_0 \supseteq 0$$

where $\overline{X} = X \otimes_{k_1} \overline{k_1}$ and $\overline{k_1}$ is an algebraic closure of k_1 . This allow him to extend the precedent results to projective smooth surfaces [3].

The aim of this paper is to use a combination of this approach and the theory of the monodromy-weight filtration of degenerating abelian varieties on local fields explained by Yoshida in his paper [11], to study curves over two-dimensional local fields (section 3).

Let X be a projective smooth curve defined over two dimensional local field k . Let K be its function field and P denotes the set of closed points of X . For each $v \in P$, $k(v)$ denotes the

1991 *Mathematics Subject Classification.* 11G25, 14H25.

Key words and phrases. Bloch-Ogus complex, Generalised reciprocity map, Higher local fields, Curves over local fields.

residue field at $v \in P$. A finite étale covering $Z \rightarrow X$ of X is called a c.s covering, if for any closed point x of X , $x \times_X Z$ is isomorphic to a finite sum of x . We denote by $\pi_1^{c.s}(X)$ the quotient group of $\pi_1^{ab}(X)$ which classifies abelian c.s coverings of X .

To study the class field theory of the curve X , we construct the generalized reciprocity map

$$\sigma/\ell : SK_2(X)/\ell \longrightarrow \pi_1^{ab}(X)/\ell$$

where $SK_2(X)/\ell = \text{Coker} \left\{ K_3(K)/\ell \xrightarrow{\oplus \partial_v} \bigoplus_{v \in P} K_2(k(v))/\ell \right\}$ and $\tau/\ell : V(X)/\ell \longrightarrow \pi_1^{ab}(X)^{géo}/\ell$ for all ℓ prime to residual characteristic. The group $V(X)$ is defined to be the kernel of the norm map $N : SK_2(X) \longrightarrow K_2(k)$ induced by the norm map $N_{k(v)/k^x} : K_2(k(v)) \longrightarrow K_2(k)$ for all v and $\pi_1^{ab}(X)^{géo}$ by the exact sequence

$$0 \longrightarrow \pi_1^{ab}(X)^{géo} \longrightarrow \pi_1^{ab}(X) \longrightarrow \text{Gal}(k^{ab}/k) \longrightarrow 0$$

The cokernel of σ/ℓ is the quotient group of $\pi_1^{ab}(X)/\ell$ that classifies completely split coverings of X ; that is; $\pi_1^{c.s}(X)/\ell$.

We begin by proving the exactness of the Kato-Saito sequence (Proposition 4.2) :

$$\begin{aligned} 0 \longrightarrow \pi_1^{c.s}(X)/\ell &\longrightarrow H^4(K, \mathbb{Z}/\ell(3)) \\ &\longrightarrow \bigoplus_{v \in P} H^3(k(v), \mathbb{Z}/\ell(2)) \longrightarrow \mathbb{Z}/\ell \longrightarrow 0 \end{aligned}$$

To determinate the group $\pi_1^{c.s}(X)/\ell$, we need to consider a semi stable model of the curve X (see Section 5) and the weight filtration on its special fiber. In fact, we will prove in (Proposition 5.1) that $\pi_1^{c.s}(X) \otimes_{\mathbb{Q}_\ell}$ admits a quotient of type \mathbb{Q}_ℓ^r where r is the rank of the first crane of this filtration.

Now, to investigate the group $\pi_1^{ab}(X)^{géo}$, we use class field theory of two-dimensional local field and prove the vanishing of the group $H^2(k, \mathbb{Q}/\mathbb{Z})$ (theorem 3.1). This yields the isomorphism

$$\pi_1^{ab}(X)^{géo} \simeq \pi_1^{ab}(\overline{X})_{G_k}$$

Finally, by the Grothendick weight filtration on the group $\pi_1^{ab}(\overline{X})_{G_k}$ and assuming the semi-stable reduction, we obtain the structure of the group $\pi_1^{ab}(X)^{géo}$ and information about the map $\tau : V(X) \longrightarrow \pi_1^{ab}(X)^{géo}$.

Our paper is organized as follows. Section 2 is devoted to some notations. Section 3 contains the proprieties which we need concerning two-dimensional local field: duality and the vanishing of the second cohomology group. In section 4, we construct the generalized reciprocity map and study the Bloch-Ogus complex associated to X . In section 5, we investigate the group $\pi_1^{c.s}(X)$.

2. NOTATIONS

For an abelian group M , and a positive integer $n \geq 1$, M/n denotes the group M/nM .

For a scheme Z , and a sheaf \mathcal{F} over the étale site of Z , $H^i(Z, \mathcal{F})$ denotes the i -th étale cohomology group. The group $H^1(Z, \mathbb{Z}/\ell)$ is identified with the group of all continues homomorphisms $\pi_1^{ab}(Z) \longrightarrow \mathbb{Z}/\ell$. If ℓ is invertible on Z , $\mathbb{Z}/\ell(1)$ denotes the sheaf of l -th root of unity and for any integer i , we denote $\mathbb{Z}/\ell(i) = (\mathbb{Z}/\ell(1))^{\otimes i}$.

For a field L , $K_i(L)$ is the i -th Milnor group. It coincides with the i -th Quillen group for $i \leq 2$. For ℓ prime to $\text{char } L$, there is a Galois symbol

$$h_{\ell, L}^i : K_i L/\ell \longrightarrow H^i(L, \mathbb{Z}/\ell(i))$$

which is an isomorphism for $i = 0, 1, 2$ ($i = 2$ is Merkur'jev-Suslin).

A local field k is said to be n -dimensional *local* if there exists the following sequence of fields k_i ($1 \leq i \leq n$) such that

- (i) each k_i is a complete discrete valuation field having k_{i-1} as the residue field of the valuation ring O_{k_i} of k_i , and
- (ii) k_0 is a finite field.

For such a field, and for ℓ prime to $\text{Char}(k)$, the well-known isomorphism

$$(3.1) \quad H^{n+1}(k, \mathbb{Z}/\ell(n)) \simeq \mathbb{Z}/\ell \quad (3.1)$$

and for each $i \in \{0, \dots, n+1\}$ a perfect duality

$$(3.2) \quad H^i(k, \mathbb{Z}/\ell(j)) \times H^{n+1-i}(k, \mathbb{Z}/\ell(n-j)) \longrightarrow H^{n+1}(k, \mathbb{Z}/\ell(n)) \simeq \mathbb{Z}/\ell \quad (3.2)$$

hold. 8 The class field theory for such fields is summarized as follows: There is a map $h : K_2(k) \longrightarrow \text{Gal}(k^{ab}/k)$ which generalizes the classical reciprocity map for usually local fields. This map induces an isomorphism $K_2(k)/N_{L/k}K_2(L) \simeq \text{Gal}(L/k)$ for each finite abelian extension L of k . Furthermore, the canonical pairing

$$(3.3) \quad H^1(k, \mathbb{Q}_l/\mathbb{Z}_l) \times K_2(k) \longrightarrow H^3(k, \mathbb{Q}_l/\mathbb{Z}_l(2)) \simeq \mathbb{Q}_l/\mathbb{Z}_l$$

induces an injective homomorphism

$$(3.4) \quad H^1(k, \mathbb{Q}_l/\mathbb{Z}_l) \longrightarrow \text{Hom}(K_2(k), \mathbb{Q}_l/\mathbb{Z}_l)$$

It is well-known that the group $H^2(M, \mathbb{Q}/\mathbb{Z})$ vanishes when M is a finite field or usually local field. Next, we prove the same result for two-dimensional local field

Theorem 3.1. *If k is a two-dimensional local field of characteristic zero, then the group $H^2(k, \mathbb{Q}/\mathbb{Z})$ vanishes.*

Proof. We proceed as in the proof of theorem 4 of [10]. It is enough to prove that $H^2(k, \mathbb{Q}_l/\mathbb{Z}_l)$ vanishes for all l and when k contains the group μ_l of l -th roots of unity. For this, we prove that multiplication by l is injective. That is, we have to show that the coboundary map

$$H^1(k, \mathbb{Q}_l/\mathbb{Z}_l) \xrightarrow{\delta} H^2(k, \mathbb{Z}/l\mathbb{Z})$$

is injective.

By assumption on k , we have

$$H^2(k, \mathbb{Z}/l\mathbb{Z}) \simeq H^2(k, \mu_l) \simeq \mathbb{Z}/\ell$$

The last isomorphism is well-known for one-dimensional local field and was generalized to non archimedean and locally compact fields by Shatz in [6]. The proof is now reduced to the fact that $\delta \neq 0$;

By class field theory of two dimensional local field, the cohomology group $H^1(k, \mathbb{Q}_l/\mathbb{Z}_l)$ may be identified with the group of continuous homomorphisms $K_2(k) \xrightarrow{\Phi} \mathbb{Q}_l/\mathbb{Z}_l$.

Now, $\delta(\Phi) = 0$ if and only if Φ is a l -th power, and Φ is a l -th power if and only if Φ is trivial on μ_l . Thus, it is sufficient to construct an homomorphism $K_2(k) \longrightarrow \mathbb{Q}_l/\mathbb{Z}_l$ which is non trivial on μ_l .

Let i be the maximal natural number such that k contains a primitive l^i -th root of unity. Then, the image ξ of a primitive l^i -th root of unity under the composite map

$$k^x/k^{xl} \simeq H^1(k, \mu_l) \simeq H^1(k, \mathbb{Z}/l\mathbb{Z}) \longrightarrow H^1(k, \mathbb{Q}_l/\mathbb{Z}_l)$$

is not zero. Thus, the injectivity of the map

$$H^1(k, \mathbb{Q}_l/\mathbb{Z}_l) \longrightarrow \text{Hom}(K_2(k), \mathbb{Q}_l/\mathbb{Z}_l)$$

gives rise to a character which is non trivial on μ_l . □

4. CURVES OVER TWO DIMENSIONAL LOCAL FIELD

Let k be a two dimensional local field of characteristic zero and X a smooth projective curve defined over k .

We recall that we denote:

$K = K(X)$ its function field,

P : set of closed points of X , and for $v \in P$,

$k(v)$: the residue field at $v \in P$

The residue field of k is one-dimensional local field. It is denoted by k_1

Let $\mathcal{H}^n(\mathbb{Z}/\ell(3))$, $n \geq 1$, the Zariskien sheaf associated to the presheaf $U \longrightarrow H^n(U, \mathbb{Z}/\ell(3))$.

Its cohomology is calculated by the Bloch-Ogus resolution. So, we have the two exact sequences:

$$(4.1) \quad H^3(K, \mathbb{Z}/\ell(3)) \longrightarrow \bigoplus_{v \in P} H^2(k(v), \mathbb{Z}/\ell(2)) \longrightarrow H^1(X_{Zar}, \mathcal{H}^3(\mathbb{Z}/\ell(3))) \longrightarrow 0$$

$$(4.2) \quad 0 \longrightarrow H^0(X_{Zar}, \mathcal{H}^4(\mathbb{Z}/\ell(3))) \longrightarrow H^4(K, \mathbb{Z}/\ell(3)) \longrightarrow \bigoplus_{v \in P} H^3(k(v), \mathbb{Z}/\ell(2))$$

4.1. The reciprocity map. We introduce the group $SK_2(X)/\ell$:

$$SK_2(X)/\ell = \text{Coker} \left\{ K_3(K)/\ell \xrightarrow{\oplus \partial_v} \bigoplus_{v \in P} K_2(k(v))/\ell \right\}$$

where $\partial_v : K_3(K) \longrightarrow K_2(k(v))$ is the boundary map in K-Theory. It will play an important role in class field theory for X as pointed out by Saito in the introduction of [8]. In this section, we construct a map

$$\sigma/\ell : SK_2(X)/\ell \longrightarrow \pi_1^{ab}(X)/\ell$$

which describe the class field theory of X .

By definition of $SK_2(X)/\ell$, we have the exact sequence

$$K_3(K)/\ell \longrightarrow \bigoplus_{v \in P} K_2(k(v))/\ell \longrightarrow SK_2(X)/\ell \longrightarrow 0$$

On the other hand, it is known that the following diagram is commutative:

$$\begin{array}{ccc} K_3(K)/\ell & \longrightarrow & \bigoplus_{v \in P} K_2(k(v))/\ell \\ \downarrow h^3 & & \downarrow h^2 \\ H^3(K, \mathbb{Z}/\ell(3)) & \longrightarrow & \bigoplus_{v \in P} H^2(k(v), \mathbb{Z}/\ell(2)) \end{array}$$

where h^2, h^3 are the Galois symbols. This yields the existence of a morphism

$$h : SK_2(X)/\ell \longrightarrow H^1(X_{Zar}, \mathcal{H}^3(\mathbb{Z}/\ell(2)))$$

taking in account the exact sequence (4.1). This morphism fit in the following commutative diagram

$$\begin{array}{ccccccc}
0 \longrightarrow & K_3(K)/\ell & \longrightarrow & \bigoplus_{v \in P} K_2(k(v))/\ell & \longrightarrow & SK_2(X)/\ell & \longrightarrow 0 \\
& \downarrow h^3 & & \downarrow h^2 & & \downarrow h & \\
0 \longrightarrow & H^3(K, \mathbb{Z}/\ell(2)) & \longrightarrow & \bigoplus_{v \in P} H^2(k(v), \mathbb{Z}/\ell(2)) & \longrightarrow & H^1(X_{Zar}, \mathcal{H}^3(\mathbb{Z}/\ell(2))) & \longrightarrow 0
\end{array}$$

By Merkur'jev-Suslin, the map h^2 is an isomorphism, which imply that h is surjective. On the other hand the spectral sequence

$$H^p(X_{Zar}, \mathcal{H}^q(\mathbb{Z}/\ell(3))) \Rightarrow H^{p+q}(X, \mathbb{Z}/\ell(3))$$

induces the exact sequence

$$\begin{aligned}
(4.3) \quad 0 \longrightarrow & H^1(X_{Zar}, \mathcal{H}^3(\mathbb{Z}/\ell(3))) \xrightarrow{e} H^4(X, \mathbb{Z}/\ell(3)) \\
& \longrightarrow H^0(X_{Zar}, \mathcal{H}^4(\mathbb{Z}/\ell(3))) \longrightarrow H^2(X_{Zar}, \mathcal{H}^3(\mathbb{Z}/\ell(3))) = 0
\end{aligned}$$

Composing h and e , we get the map

$$SK_2(X)/\ell \longrightarrow H^4(X, \mathbb{Z}/\ell(3))$$

Finally the group $H^4(X, \mathbb{Z}/\ell(3))$ is identified to the group $\pi_1^{ab}(X)/\ell$ by the duality [4, II, th 2.1]

$$H^4(X, \mathbb{Z}/\ell(3)) \otimes H^1(X, \mathbb{Z}/\ell) \longrightarrow H^5(X, \mathbb{Z}/\ell(3)) \simeq H^3(k, \mathbb{Z}/\ell(2)) \simeq \mathbb{Z}/\ell$$

Hence, we obtain the map

$$\sigma/\ell : SK_2(X)/\ell \longrightarrow \pi_1^{ab}(X)/\ell$$

Remark 4.1. By the exact sequence (4.2) the group $H^0(X_{Zar}, \mathcal{H}^4(\mathbb{Z}/\ell(3)))$ coincides with the kernel of the map

$$H^4(K, \mathbb{Z}/\ell(3)) \longrightarrow \bigoplus_{v \in P} H^3(k(v), \mathbb{Z}/\ell(2))$$

and by localization in étale cohomology

$$\bigoplus_{v \in P} H^2(k(v), \mathbb{Z}/\ell(2)) \longrightarrow H^4(X, \mathbb{Z}/\ell(3)) \longrightarrow H^4(K, \mathbb{Z}/\ell(3)) \longrightarrow \bigoplus_{v \in P} H^3(k(v), \mathbb{Z}/\ell(2))$$

and taking in account (4.3), we see that $H^1(X_{Zar}, \mathcal{H}^4(\mathbb{Z}/\ell(3)))$ is the cokernel of the Gysin map

$$\bigoplus_{v \in P} H^2(k(v), \mathbb{Z}/\ell(2)) \xrightarrow{g} H^4(X, \mathbb{Z}/\ell(3))$$

and consequently the morphism g factorize through $H^1(X_{Zar}, \mathcal{H}^4(\mathbb{Z}/\ell(3)))$

$$\begin{array}{ccc}
\bigoplus_{v \in P} H^2(k(v), \mathbb{Z}/\ell(2)) & \xrightarrow{g} & H^4(X, \mathbb{Z}/\ell(3)) \\
\searrow & & \nearrow \\
& H^1(X_{Zar}, \mathcal{H}^4(\mathbb{Z}/\ell(3))) &
\end{array}$$

Then, we deduce the following commutative diagram

$$\begin{array}{ccccc}
K_3(K)/\ell & \rightarrow \bigoplus_{v \in P} & K_2(k(v))/\ell & \rightarrow & SK_2(X)/\ell \rightarrow 0 \\
\downarrow h^3 & & \downarrow h^2 & & \downarrow h \\
H^3(K, \mathbb{Z}/\ell(3)) & \rightarrow \bigoplus_{v \in P} & H^2(k(v), \mathbb{Z}/\ell(2)) & \rightarrow & H^1(X_{Zar}, \mathcal{H}^4(\mathbb{Z}/\ell(3))) \rightarrow 0 \\
& & \downarrow g & \swarrow e & \\
& & \pi_1^{ab}(X)/\ell = H^4(X, \mathbb{Z}/\ell(3)) & &
\end{array}$$

The surjectivity of the map h implies that the cokernel of

$$\sigma/\ell : SK_2(X)/\ell \rightarrow \pi_1^{ab}(X)/\ell$$

coincides with the cokernel of e which is $H^0(X_{Zar}, \mathcal{H}^4(\mathbb{Z}/\ell(3)))$. Hence $\text{Coker } \sigma/\ell$ is the dual of the kernel of the map

$$(4.4) \quad H^1(X, \mathbb{Z}/\ell) \rightarrow \prod_{v \in P} H^1(k(v), \mathbb{Z}/\ell)$$

4.2. The Kato-Saito exact sequence.

Definition 4.2. Let Z be a Noetherian scheme. A finite etale covering $f : W \rightarrow Z$ is called a c.s covering if for any closed point z of Z , $z \times_Z W$ is isomorphic to a finite scheme-theoretic sum of copies of z . We denote $\pi_1^{c.s}(Z)$ the quotient group of $\pi_1^{ab}(Z)$ which classifies abelian c.s coverings of Z .

Hence, the group $\pi_1^{c.s}(X)/\ell$ is the dual of the kernel of the map

$$(4.4) \quad H^1(X, \mathbb{Z}/\ell) \rightarrow \prod_{v \in P} H^1(k(v), \mathbb{Z}/\ell)$$

as in [8, section 2, definition and sentence just below]. Now, we are able to calculate the homologies of the Bloch-Ogus complex associated X .

Generalizing [9, Theorem 7], we obtain :

Proposition 4.3. *Let X be a projective smooth curve defined over k . Then for all ℓ , we have the following exact sequence*

$$\begin{array}{ccccccc}
0 & \longrightarrow & \pi_1^{c.s}(X)/\ell & \longrightarrow & H^4(K, \mathbb{Z}/\ell(3)) & & \\
& & & \longrightarrow & \bigoplus_{v \in P} H^3(k(v), \mathbb{Z}/\ell(2)) & \longrightarrow & \mathbb{Z}/\ell \longrightarrow 0.
\end{array}$$

Proof. Consider the localization sequence on X

$$\begin{array}{ccccccc}
\bigoplus_{v \in P} H^2(k(v), \mathbb{Z}/\ell(2)) & \xrightarrow{g} & H^4(X, \mathbb{Z}/\ell(3)) & \longrightarrow & H^4(K, \mathbb{Z}/\ell(3)) & & \\
\longrightarrow & \bigoplus_{v \in P} H^3(k(v), \mathbb{Z}/\ell(2)) & \longrightarrow & H^5(X, \mathbb{Z}/\ell(3)) & \longrightarrow & 0 &
\end{array}$$

We know that the cokernel of the Gysin map g coincides with $\pi_1^{c.s}(X)/\ell$ and we use the isomorphism $H^5(X, \mathbb{Z}/\ell(3)) \simeq \mathbb{Z}/\ell$ (4.4). \square

In his paper [8], Saito don't prove the p -primary part in the char $k = p > 0$ case. This case was developed by Yoshida in [11]. His method is based on the theory of monodromy-weight filtration of degenerating abelian varieties on local fields. In this work, we use this approach to investigate the group $\pi_1^{c.s.}(X)$. As mentioned by Yoshida in [11, section 2] Grothendieck's theory of monodromy-weight filtration on Tate module of abelian varieties are valid where the residue field is arbitrary perfect field

We assume the semi-stable reduction and choose a regular model \mathcal{X} of X over $\text{Spec} O_k$, by which we mean a two dimensional regular scheme with a proper birational morphism $f : \mathcal{X} \rightarrow \text{Spec} O_k$ such that $\mathcal{X} \otimes_{O_k} k \simeq X$ and if \mathcal{X}_s designates the special fiber $\mathcal{X} \otimes_{O_k} k_1$, then $Y = (\mathcal{X}_s)_{\text{red}}$ is a curve defined over the residue field k_1 such that any irreducible component of Y is regular and it has ordinary double points as singularity.

Let $\bar{Y} = Y \otimes_{k_1} \bar{k}_1$, where \bar{k}_1 is an algebraic closure of k_1 and $\bar{Y}^{[p]} = \bigsqcup_{i_1 < \dots < i_p} \bar{Y}_{i_1} \cap \bar{Y}_{i_2} \cap \dots \cap \bar{Y}_{i_p}$

, $(\bar{Y})_{i \in I}$ = collection of irreducible components of \bar{Y} .

Let $|\bar{\Gamma}|$ be a realization of the dual graph $\bar{\Gamma}$, then the group $H^1(|\bar{\Gamma}|, \mathbb{Q}_l)$ coincides with the group $W_0(H^1(\bar{Y}, \mathbb{Q}_l))$ constituted of elements of weight 0 for the filtration

$$H^1(\bar{Y}, \mathbb{Q}_\ell) = W_1 \supseteq W_0 \supseteq 0$$

of $H^1(\bar{Y}, \mathbb{Q}_\ell)$ deduced from the spectral sequence

$$E_1^{p,q} = H^q(\bar{Y}^{[p]}, \mathbb{Q}_\ell) \implies H^{p+q}(\bar{Y}, \mathbb{Q}_\ell)$$

For details see [2], [3] and [5]

Now, if we assume further that the irreducible components and double points of \bar{Y} are defined over k_1 , then the dual graph $\bar{\Gamma}$ of \bar{Y} go down to k_1 and we obtain the injection

$$W_0(H^1(\bar{Y}, \mathbb{Q}_l)) \subseteq H^1(Y, \mathbb{Q}_l) \hookrightarrow H^1(X, \mathbb{Q}_l)$$

Proposition 5.1. *The group $\pi_1^{c.s.}(X) \otimes \mathbb{Q}_l$ admits a quotient of type \mathbb{Q}_l^r , where r is the \mathbb{Q}_l -rank of the group $H^1(|\bar{\Gamma}|, \mathbb{Q}_l)$*

Proof. We know (4.5) that $\pi_1^{c.s.}(X) \otimes \mathbb{Q}_l$ is the dual of the kernel of the map

$$\alpha : H^1(X, \mathbb{Q}_l) \longrightarrow \prod_{v \in P} H^1(k(v), \mathbb{Q}_l)$$

We will prove that $W_0(H^1(\bar{Y}, \mathbb{Q}_l)) \subseteq \text{Ker} \alpha$. The group $W_0 = W_0(H^1(\bar{Y}, \mathbb{Q}_l))$ is calculated as the homologie of the complex

$$H^0(\bar{Y}^{[0]}, \mathbb{Q}_\ell) \longrightarrow H^0(\bar{Y}^{[1]}, \mathbb{Q}_\ell) \longrightarrow 0$$

Hence $W_0 = H^0(\bar{Y}^{[1]}, \mathbb{Q}_\ell) / \text{Im}\{H^0(\bar{Y}^{[0]}, \mathbb{Q}_\ell) \longrightarrow H^0(\bar{Y}^{[1]}, \mathbb{Q}_\ell)\}$. Thus, it suffices to prove the vanishing of the composing map

$$H^0(\bar{Y}^{[1]}, \mathbb{Q}_\ell) \longrightarrow W_0 \subseteq H^1(Y, \mathbb{Q}_l) \hookrightarrow H^1(X, \mathbb{Q}_l) \longrightarrow H^1(k(v), \mathbb{Q}_l)$$

for all $v \in P$.

Let z_v be the 0-cycle in \bar{Y} obtained by specializing v , which induces a map $z_v^{[1]} \longrightarrow \bar{Y}^{[1]}$. Consequently, the map $H^0(\bar{Y}^{[1]}, \mathbb{Q}_\ell) \longrightarrow H^1(k(v), \mathbb{Q}_l)$ factors as follows

$$\begin{array}{ccc}
H^0(\overline{Y}^{[1]}, \mathbb{Q}_\ell) & \longrightarrow & H^1(k(v), \mathbb{Q}_\ell) \\
\searrow & & \nearrow \\
& H^0(z_v^{[1]}, \mathbb{Q}_\ell) &
\end{array}$$

But the trace $z_v^{[1]}$ of $\overline{Y}^{[1]}$ on z_v is empty. This implies the vanishing of $H^0(z_v^{[1]}, \mathbb{Q}_\ell)$. \square

Let $V(X)$ be the kernel of the norm map $N : SK_2(X) \longrightarrow K_2(k)$ induced by the norm map $N_{k(v)/k^x} : K_2(k(v)) \longrightarrow K_2(k)$ for all v . Then, we obtain a map $\tau/l : V(X)/\ell \longrightarrow \pi_1^{ab}(X)^{géo}/\ell$ and a commutative diagram

$$\begin{array}{ccccc}
V(X)/\ell & \longrightarrow & SK_2(X)/\ell & \longrightarrow & K_2(k)/\ell \\
\downarrow \tau/l & & \downarrow \sigma/\ell & & \downarrow h/l \\
\pi_1^{ab}(X)^{géo}/\ell & \longrightarrow & \pi_1^{ab}(X)/\ell & \longrightarrow & Gal(k^{ab}/k)/l
\end{array}$$

where the map $h/l : K_2(k)/\ell \longrightarrow Gal(k^{ab}/k)/l$ is the one obtained by class field theory of k (section 3). From this diagram we see that the group $\text{Coker } \tau/l$ is isomorphic to the group $\text{Coker } \sigma/\ell$. Next, we investigate the map τ/l .

We begin by the following result which is a consequence of the structure of the two-dimensional local field k

Lemma 5.2. *There is an isomorphism*

$$\pi_1^{ab}(X)^{géo} \simeq \pi_1^{ab}(\overline{X})_{G_k},$$

where $\pi_1^{ab}(\overline{X})_{G_k}$ is the group of coinvariants under $G_k = Gal(k^{ab}/k)$.

Proof. As in the proof of Lemma 4.3 of [11], this is an immediate consequence of (Theorem 3.1). \square

Finally, we are able to deduce the structure of the group $\pi_1^{ab}(X)^{géo}$

Theorem 5.3. *The group $\pi_1^{ab}(X)^{géo} \otimes \mathbb{Q}_l$ is isomorphic to $\widehat{\mathbb{Q}_l}^r$ and the map $\tau : V(X) \longrightarrow \pi_1^{ab}(X)^{géo}$ is a surjection onto $(\pi_1^{ab}(X)^{géo})_{tor}$.*

Proof. By the preceding lemma, we have the isomorphism $\pi_1^{ab}(X)^{géo} \simeq \pi_1^{ab}(\overline{X})_{G_k}$. On the other hand the group $\pi_1^{ab}(\overline{X})_{G_k} \otimes \mathbb{Q}_\ell$ admits the filtration [12, Lemma 4.1 and section 2]

$$W_0(\pi_1^{ab}(\overline{X})_{G_k} \otimes \mathbb{Q}_l) = \pi_1^{ab}(\overline{X})_{G_k} \otimes \mathbb{Q}_l \supseteq W_{-1}(\pi_1^{ab}(\overline{X})_{G_k} \otimes \mathbb{Q}_l) \supseteq W_{-2}(\pi_1^{ab}(\overline{X})_{G_k} \otimes \mathbb{Q}_l)$$

But, by assumption; the curve X admits a semi-stable reduction, then the group $Gr_0(\pi_1^{ab}(\overline{X})_{G_k} \otimes \mathbb{Q}_l) = W_0(\pi_1^{ab}(\overline{X})_{G_k} \otimes \mathbb{Q}_l)/W_{-1}(\pi_1^{ab}(\overline{X})_{G_k} \otimes \mathbb{Q}_l)$ has the following structure

$$0 \longrightarrow Gr_0(\pi_1^{ab}(\overline{X})_{G_k} \otimes \mathbb{Q}_l)_{tor} \longrightarrow Gr_0(\pi_1^{ab}(\overline{X})_{G_k} \otimes \mathbb{Q}_l) \longrightarrow \widehat{\mathbb{Q}_l}^{r'} \longrightarrow 0$$

where r' is the k -rank of X . This is confirmed by Yoshida [11, section 2], independently of the finitude of the residue field of k considered in his paper. The integer r' is equal to the integer $r = H^1(\overline{\Gamma}, \mathbb{Q}_l) = H^1(\Gamma, \mathbb{Q}_l)$ by assuming that the irreducible components and double points of \overline{Y} are defined over k_1 ,

On the other hand, the exact sequence

$$0 \longrightarrow W_{-1}(\pi_1^{ab}(\overline{X})_{G_k}) \longrightarrow \pi_1^{ab}(\overline{X})_{G_k} \longrightarrow Gr_0(\pi_1^{ab}(\overline{X})_{G_k}) \longrightarrow 0$$

and (Proposition 5.1) allow us to conclude that the group $W_{-1}(\pi_1^{ab}(\overline{X})_{G_k})$ is finite and the map $\tau : V(X) \longrightarrow \pi_1^{ab}(X)^{géo}$ is a surjection onto $(\pi_1^{ab}(X)^{géo})_{tor}$ as established by Yoshida [11] \square

Remark 5.4. If we apply the same method of Saito to study curves over two-dimensional local fields, we need class field theory of two-dimensional local ring having one-dimensional local field as residue field. This is done by myself in [1]. Hence, one can follow Saito's method to obtain the same results.

REFERENCES

- [1] Draouil, B. *Cohomological Hasse principle for the ring $\mathbb{F}_p((t))[[X, Y]]$* , Bull. Belg. Math. Soc. Simon Stevin 11, no. 2 (2004), pp 181–190
- [2] Draouil, B., Douai, J. C. *Sur l'arithmétique des anneaux locaux de dimension 2 et 3*, Journal of Algebra **213** (1999), pp 499–512.
- [3] Douai, J. C. *Monodromie et Arithmétique des Surfaces* Birkhauser, Février (1993)
- [4] Douai, J. C. *Le théorème de Tate-Poitou pour le corps des foncitions définies sur les corps locaux de dim N*, Journal of Algebra Vol 125 N° II August **15**, (1989) ,pp 181–196.
- [5] Morrisson, D. R. *The Clemens-Schmid exact sequence and applications*, in Annals of Mathematics Studies Vol. **106**, , Princeton Univ. Press, Princeton NJ, pp 101–119
- [6] Shatz S. S. *Cohomology of Artinian group schemes over local fields*, Annals of Maths (2) **88** (1968), pp 492–517
- [7] Saito, S. *Class field Theory for two-dimensional local rings Galois groups and their representations*, Kinokuniya-North Holland Amsterdam, vol 12 (1987), pp 343–373
- [8] Saito, S. *Class field theory for curves over local fields*, Journal of Number theory 21 (1985), pp 44–80. 8
- [9] Saito, S. *Some observations on motivic cohomology of arithmetic schemes*. Invent.math. 98 (1989), pp 371–404.
- [10] Serre, J. P. *Modular forms of weight one and Galois representations*, Algebraic Number Thory, Academic Press, (1977), pp 193–268.
- [11] Yoshida, T. *Finitness theorems in the class field theory of varieties over local fields*, Journal of Number Theory **101** (2003), pp 138–150.

DÉPARTEMENT DE MATHÉMATIQUES., FACULTÉ DES SCIENCES DE BIZERTE 7021, ZARZOUNA BIZERTE.
E-mail address: Belgacem.Draouil@fsb.rnu.tn